STABILITY OF A NONSTATIONARY ROUND JET OF AN IDEAL INCOMPRESSIBLE FLUID

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The stability of transient flow in a cylinder of an ideal incompressible fluid with a free boundary is studied. There are 20 different cases of the behavior of small disturbances as a function of the parameters of the problem. In particular, if surface tension is not taken into account a round jet is stable with respect to axially symmetrical disturbances, but the introduction of capillary forces leads to a strong instability.

The stability of nonstationary flow in a strip of ideal incompressible fluid with a linear velocity field was examined in [1]. This flow proved to be unstable when surface tension was not taken into account, while the introduction of surface tension stabilizes the flow.

The general problem of the stability of nonstationary currents of an ideal incompressible fluid with a free boundary was formulated in [2] by L. V. Ovsyannikov. Examples demonstrating the difficulty of studying instability are also presented there.

The case when the free boundary has the form of an ellipse was analyzed in [3]. It develops that if a standard in L_2 of the disturbance potential is taken as the measure of the stability, the flow is stable. But if the stability is judged by the deviation of the free boundary from its undisturbed state, the movement is unstable.

The stability of nonstationary flow in a cylinder of ideal incompressible fluid with a free boundary is examined here.

1. Formulation of Problem

A region Ω, filled by an ideal incompressible fluid, is represented by the cylinder $0 \leq z \leq h$, $x^2 + y^2 \leq R^2$. Here, x, y, and z are Euler coordinates. The cylinder is bounded by two impenetrable walls. The lateral surface G is free and the pressure undergoes a discontinuity at G: $p - p_1 = \sigma/R$, $p_1 = \text{const}$, and σ is the coefficient of surface tension. The constant p_1 is taken as equal to zero without restriction. At the time $t = 0$ one of the walls suddenly begins to move with a velocity $V = \text{const}$, while the other remains stationary.

The basic solution in the Lagrange coordinates $\varkappa$, η, ζ has the form [2]

$$x = \alpha\xi, \; y = \alpha\eta, \quad z = \alpha^{-2}\zeta$$
$$p = {}^1/_2\alpha\alpha'' (R^2 - \xi^2 - \eta^2) + \sigma / R\alpha \qquad (1.1)$$
$$\alpha = (1 + \varkappa t)^{-1/2}, \quad \varkappa = V / h$$

A prime denotes differentiation with respect to t. For all $t \geq 0$ the free boundary is the round cylinder $0 \leq J \leq h$, $\xi^2 + \eta^2 = R^2$. With an increase in t the cylinder Ω contracts toward the axis $x = y = 0$ for $\varkappa > 0$, while for $\varkappa < 0$ it expands to infinity in a time $t^* = -1/\varkappa$.

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Let us examine another solution in the region Ω, but with converted initial functions

$$\varphi_0^*(\xi) = \varphi_0(\xi) + \Phi_0(\xi), \quad \Delta\Phi_0 = 0 \tag{1.2}$$

We set

$$\mathbf{x}^* = \mathbf{x} + \mathbf{X}, \quad \varphi^* = \varphi + \mathbf{x}_t\mathbf{X} + \Phi$$

Then the problem of the evolution of disturbances in the basic solution, which takes place under the influence of a small disturbance in the initial function (1.2), in the linear approximation has the form [2]

$$\operatorname{div} M^{-1}M^{*-1}\nabla\Phi = 0 \quad (\xi \in \Omega) \tag{1.3}$$

$$\Phi_t + P + \mathbf{x}_{tt}M\int_0^t M^{-1}M^{*-1}\nabla\Phi\,dt = 0 \quad (\xi \in \Gamma) \tag{1.4}$$

$$\Phi = \Phi_0(\xi), \quad \Delta\Phi_0 = 0 \quad (t = 0) \tag{1.5}$$

Here P is the linear section of increases in the function $\sigma(R_1^{-1} + R_2^{-1})$; R_1 and R_2 are the principal radii of curvature of the normal cross sections at the given point of the disturbed surface; M is the Jacobian transformation matrix $\mathbf{x} = \mathbf{x}(\xi, t)$; and from (1.1) $M = M^* = \operatorname{diag}(\alpha, \alpha, \alpha^{-2})$.

We change to the cylindrical coordinates ρ, θ, ζ through the equations

$$\xi = \rho\cos\theta, \quad \eta = \rho\sin\theta, \quad \zeta = \zeta$$

Then Eq. (1.3) gives

$$\Phi_{\rho\rho} + \rho^{-1}\Phi_\rho + \rho^{-2}\Phi_{\theta\theta} + \alpha^6\Phi_{\zeta\zeta} = 0 \quad (\rho < R) \tag{1.6}$$

From (1.4) we obtain

$$\Phi_t + P + \alpha\alpha''\rho\int_0^t \frac{1}{\alpha^2}\Phi_\rho dt = 0 \quad (\rho = R) \tag{1.7}$$

We shall describe the behavior of the disturbed boundary of the cylinder by the normal component of vector $\mathbf{X}$. According to [2],

$$\mathbf{X} = M\int_0^t M^{-1}M^{*-1}\nabla\Phi dt$$

Then we obtain for the normal component

$$\eta^* = \alpha\int_0^t \frac{1}{\alpha^2}\Phi_\rho\,dt \tag{1.8}$$

The equation of the disturbed surface in cylindrical (Euler) coordinates has the form

$$\rho = R\alpha + \eta^*(\theta, z, t)$$

An obvious expression for $R_1^{-1} + R_2^{-1}$ can be found from the differential geometry according to well-known equations. Then, in view of the smallness of η^* and its derivatives, we carry out an expansion with respect to it, and we retain in this expansion the terms linear with respect to η^* and its derivatives

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R\alpha} - \frac{\eta^*}{R^2\alpha^2} - \frac{\eta_{\theta\theta}^*}{R^2\alpha^2} - \eta_{zz}^* + \ldots$$

In Lagrange coordinates

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R\alpha} - \frac{\eta^*}{R^2\alpha^2} - \frac{\eta_{\theta\theta}^*}{R^2\alpha^2} - \alpha^4\eta_{\zeta\zeta}^* + \ldots \tag{1.9}$$

The first term of the expansion corresponds to a purely cylindrical surface.

2. Separation of Variables

We shall introduce the dimensionless variables through the equations

$$\rho_1 \to \frac{\rho}{R}\alpha^3, \quad \alpha \to t, \quad \zeta_1 \to \frac{\zeta}{h}, \quad \Phi_1 \to (Rh\varkappa)^{-1}\Phi \tag{2.1}$$

Then from (1.6) and (1.7), taking (1.9) into account, we obtain (the indices are omitted for simplicity)

$$\Phi_{\rho\rho} + \frac{1}{\rho}\Phi_\rho + \frac{1}{\rho^2}\Phi_{\theta\theta} + \frac{R^2}{h^2}\Phi_{\zeta\zeta} = 0 \quad (\rho < \alpha^3) \tag{2.2}$$

$$\Phi_\alpha + 3\alpha^2\Phi_\rho - \frac{2\sigma}{\varkappa^2 Rh\alpha^3}\left(\frac{2h}{R^2\alpha}\int_1^\alpha \frac{1}{\alpha^2}\Phi_\rho\, d\alpha + \frac{2\alpha^5}{h}\int_1^\alpha \frac{1}{\alpha^2}\Phi_{\rho\zeta\zeta}\, d\alpha + \right.$$

$$\left. + \frac{2h}{R^2\alpha}\int_1^\alpha \frac{1}{\alpha^2}\Phi_{\rho\theta\theta}\, d\alpha\right) + 3\alpha^3\int_1^\alpha \frac{1}{\alpha^2}\Phi_\rho\, d\alpha = 0 \quad (\rho = \alpha^3) \tag{2.3}$$

From (1.8) it follows that

$$\eta^* = -2h\alpha\int_1^\alpha \frac{1}{\alpha^2}\Phi_\rho\, d\alpha \tag{2.4}$$

All the variables in Eqs. (2.2) and (2.3) are isolated, which means that in the problem of stability one can confine the investigation to partial solutions of the type

$$\Phi = T(\alpha)A(\rho)\exp i(k\zeta + \lambda\theta) \tag{2.5}$$

Here the wave number $k = n\pi$; $n = 0, 1, 2, \ldots.$ The spectral mode of λ is an integer. It follows from (2.2) that $A(\rho) = I_\lambda(kRh^{-1}\rho)$ is a modified Bessel function of the first kind.

Let us introduce the dimensionless parameters

$$R/h = \beta, \ \sigma/R^3\varkappa^2 = \gamma \tag{2.6}$$

$$A_1 = TI_\lambda, \quad A_2 = \int_1^\alpha \frac{1}{\alpha^2} TI_\lambda'\, d\alpha, \quad I_\lambda' = \frac{d}{d\rho} I_\lambda(kRh^{-1}\rho)\,|_{\rho=\alpha^3}$$

Using the notations (2.6) we finally obtain from (2.3) a system of linear differential equations of first order:

$$dA_1/d\alpha = -[4\gamma(\lambda^2 - 1)\alpha^{-4} + 4\gamma\beta^2k^2\alpha^2 + 3\alpha^3]A_2$$
$$\frac{dA_2}{d\alpha} = \frac{I_\lambda'}{\alpha^2 I_\lambda}A_1 \tag{2.7}$$

It follows from (2.4) that the behavior of η^* is described by the function A_2. Evidently

$$\eta^* = -2h\alpha\int_1^\alpha \frac{1}{\alpha^2}\Phi_\rho\, d\alpha = -2h\alpha A_2 \exp i(k\zeta + \lambda\theta) \tag{2.8}$$

3. Asymptotic Nature of Solutions

There are 20 different cases of the behavior of A_2 in dependence on the parameters β, γ, k, and λ of the problem. Here we shall present the asymptotic nature of A_2 (see, for example, [4]).

Suppose a cylinder is contracting toward the axis x = y = 0, and the parameters γ and k are not simultaneously equal to zero. Then

$$A_2 \sim \alpha^{5/4}\left[c_1 \operatorname{ch}\left(2\beta k \sqrt{2\gamma}\,\alpha^{-1/2}\right) + c_2 \operatorname{sh}\left(2\beta k \sqrt{2\gamma}\,\alpha^{-1/2}\right)\right] \quad (\lambda = 0) \tag{3.1}$$

$$A_2 \sim \alpha^{-7/4}\left[c_1 \cos\left(4\beta k \sqrt{\gamma}\,\alpha^{-1/2}\right) + c_2 \sin\left(4\beta k \sqrt{\gamma}\,\alpha^{-1/2}\right)\right] \quad (\lambda = 1) \tag{3.2}$$

$$A_2 \sim \alpha^{-7/4}\left\{c_1 \cos\left[{}^4/_7 \sqrt{\lambda(\lambda^2-1)\gamma}\,\alpha^{-7/2}\right] + c_2 \sin\left[{}^4/_7 \sqrt{\lambda(\lambda^2-1)\gamma}\,\alpha^{-7/2}\right]\right\} \quad (\lambda > 1) \tag{3.3}$$

Suppose the cylinder is expanding to infinity, and γ and k are not simultaneously equal to zero as before. In this case for λ we have

$$A_2 \sim \alpha^{-5/4}\left[c_1 \cos\left({}^2/_3 \sqrt{3k\beta}\,\alpha^{3/2}\right) + c_2 \sin\left({}^2/_3 \sqrt{3k\beta}\,\alpha^{3/2}\right)\right] \tag{3.4}$$

Let us assume that $\gamma = 0$ and $k \neq 0$. Then for the expanding cylinder we have

$$A_2 \sim c_1 + c_2\alpha^2 \quad (\lambda = 0) \tag{3.5}$$

$$A_2 \sim c_1/\alpha + c_2/\alpha^3 \quad (\lambda = 1) \tag{3.6}$$

$$A_2 \sim \alpha^{-2}\left[c_1 \cos\left(\sqrt{3\lambda - 4}\ln\alpha\right) + c_2 \sin\left(\sqrt{3\lambda - 4}\ln\alpha\right)\right] \quad (\lambda > 1) \tag{3.7}$$

For contraction, for all λ we obtain

$$A_2 \sim \alpha^{-5/4}\left[c_1 \cos\left({}^2/_3 \sqrt{3k\beta}\,\alpha^{3/2}\right) + c_2 \sin\left({}^2/_3 \sqrt{3k\beta}\,\alpha^{3/2}\right)\right] \tag{3.8}$$

Let us examine a planar disturbance, which corresponds to k = 0. The function $A(\rho) = \rho^\lambda$, and because the fluid is incompressible the case of $\lambda = 0$ drops out. For expansion of the cylinder and $\gamma \neq 0$ we have

$$A_2 \sim c_1/\alpha + c_2/\alpha^3 \quad (\lambda = 1) \tag{3.9}$$

$$A_2 \sim \alpha^{-1/4}\left\{c_1 \cos\left[{}^4/_7 \sqrt{\lambda(\lambda^2-1)\gamma}\,\alpha^{-7/2}\right] + c_2 \sin\left[{}^4/_7 \sqrt{\lambda(\lambda^2-1)\gamma}\,\alpha^{-7/2}\right]\right\} \quad (\lambda > 1) \tag{3.10}$$

For contraction and $\gamma \neq 0$

$$A_2 \sim c_1/\alpha + c_2/\alpha^3 \quad (\lambda = 1) \tag{3.11}$$

$$A_2 \sim \alpha^{-2}\left[c_1 \cos\left(\sqrt{3\lambda - 4}\ln\alpha\right) + c_2 \sin\left(\sqrt{3\lambda - 4}\ln\alpha\right)\right] \quad (\lambda > 1) \tag{3.12}$$

If $\gamma = 0$ and $\alpha \to 0$ the asymptotic form coincides with (3.6) and (3.7), respectively, while for $\gamma = 0$ and $\alpha \to \infty$ it coincides with (3.11) and (3.12).

Thus, stability is observed for contraction of the jet, while for expansion the axially symmetrical disturbances become unstable with the imposition of surface tension, with the instability being of the exponential type. By calculating the coordinates of the center of mass one can verify that a shift of the center of mass in the x, z plane takes place in the case when $\lambda = 1$ and only in this case; i.e., a displacement of the jet as a whole occurs. For $\lambda > 1$ the surface tension stabilizes the flow.

It is known [5] that in the case of a stationary jet axially symmetrical disturbances are unstable for wavelengths greater than the radius of curvature of the cross section. However, in a nonstationary jet instability is observed for any wavelength.

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